Section 2 2/11/14

- Dynamic Height

Q: Again, what do the currents look like in the ocean?

A: Recall geostrophy

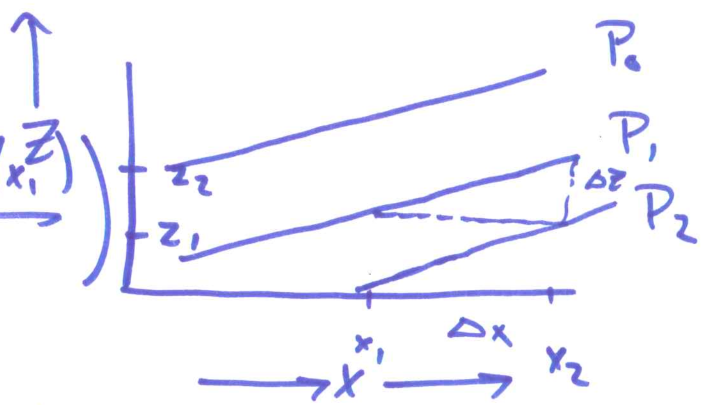
$$u_g = -\frac{1}{f\rho_0}\frac{\partial P}{\partial y} \qquad v_g = +\frac{1}{f\rho_0}\frac{\partial P}{\partial x}$$

and hydrostasy

$$\frac{\partial P}{\partial z} = -\rho g$$

Let's look at these equations along lines of constant pressure

Now (see diagram)




$$P(x_2) - P(x_1) = \Delta x \left( \frac{P(x_2)-P(x_1)}{\Delta x} \right)\Bigg|_z$$

$$= \Delta x \left(\frac{P(z_2)-P(z_1)}{\Delta z}\right)\left(\frac{\Delta z}{\Delta x}\right)\Bigg|_P \times (-1)$$

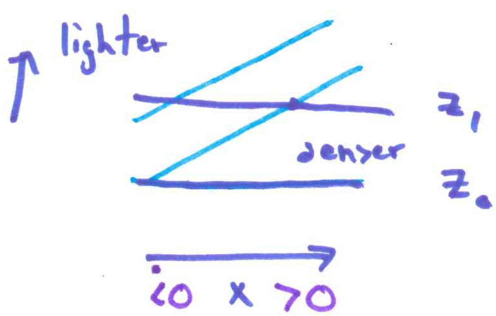
$$\Rightarrow \left.\frac{\partial P}{\partial x}\right|_z = -\frac{\partial P}{\partial z}\left.\frac{\partial z}{\partial x}\right|_P$$

Why (-1)?

MATH

Also, Change of basis requires the Jacobian.

$$J\left(\frac{x,z}{x,P}\right) = \frac{\partial z}{\partial P} < 0 \ !$$




$$\frac{\partial P}{\partial x} > 0$$

$$\frac{\partial P}{\partial z} < 0 \Rightarrow (-1)\ !$$

$$\left.\frac{\partial z}{\partial x}\right|_P > 0$$

$>0$

$$\left.\frac{\partial P}{\partial x}\right|_z = (\pm 1) \cdot \frac{\partial P}{\partial z}\frac{\partial z}{\partial x}$$

Anyways, we have

hydrostasy!

$$\left.\frac{\partial P}{\partial x}\right|_z = -\frac{\partial P}{\partial z}\left.\frac{\partial z}{\partial x}\right|_P = \underbrace{\rho g \left.\frac{\partial z}{\partial x}\right|_P}$$

Let's call this $\frac{\partial \Phi}{\partial x}$, the geo-potential

$$\Rightarrow \left.\frac{\partial P}{\partial x}\right|_z = \rho \frac{\partial \Phi}{\partial x}$$

$$\frac{\partial P}{\partial y} = -\rho \frac{\partial \Phi}{\partial y}$$

$$\Rightarrow \quad u_g = \frac{-\rho}{f\rho_0}\frac{\partial \Phi}{\partial y}$$

$$v_g = \frac{+\rho}{f\rho_0}\frac{\partial \Phi}{\partial x}$$

Assume $\frac{\rho}{\rho_0} \approx 1$ (The Boussinesq approx)

$$\Rightarrow \quad u_g = \frac{-1}{f}\frac{\partial \Phi}{\partial y}, \quad v_g = \frac{+1}{f}\frac{\partial \Phi}{\partial x}$$

But we don't know $\Phi$ <u>everywhere</u>

So differentiate!

Differentiating in pressure coordinates

$$\frac{\partial u_g}{\partial p} = \frac{-1}{f}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{-1}{f}\frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial p}\right)$$

[since we assume y lies on]
[constant p surfaces]

$$= \frac{-1}{f}\frac{\partial}{\partial y}\left(\frac{\partial}{\partial p}(gz)\right) \qquad \frac{\partial P}{\partial z} = -\rho g \text{ (recall)}$$

$$= \frac{-1}{f}\frac{\partial}{\partial y}\left(\cancel{g}\left(\frac{-1}{\rho \cancel{g}}\right)\right)$$

$$= +\frac{1}{f}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) \; ! \; \ldots \; \frac{\partial v_g}{\partial p} = \frac{-1}{f}\frac{\partial}{\partial x}\left(\frac{1}{\rho}\right)$$

## The Thermal Wind Eqn. In P-Coords

$$\frac{\partial u}{\partial p} = + \frac{1}{f}\frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) \qquad \frac{\partial v}{\partial p} = - \frac{1}{f}\frac{\partial}{\partial x}\left(1/\rho\right)$$

Lets integrate from two known pressure levels, $P_1 \to P_2$

$$\int_{P_1}^{P_2} \frac{\partial u}{\partial p}\, dp = u(P_2) - u(P_1) = \frac{1}{f} \int_{P_1}^{P_2} \frac{\partial}{\partial y}\left(\frac{1}{\rho}\right) dp$$

Let's say $\alpha \equiv 1/\rho$, the "specific volume"

$$\delta(S,T,p) = \alpha(S,T,p) - \alpha_0$$

Then $$u(P_2) - u(P_1) = \frac{1}{f}\frac{\partial}{\partial y} \int_{P_1}^{P_2} \delta\, dp$$

$$v(P_2) - v(P_1) = -\frac{1}{f}\frac{\partial}{\partial x} \int_{P_1}^{P_2} \delta\, dp$$

Define the Dynamic Height between two levels

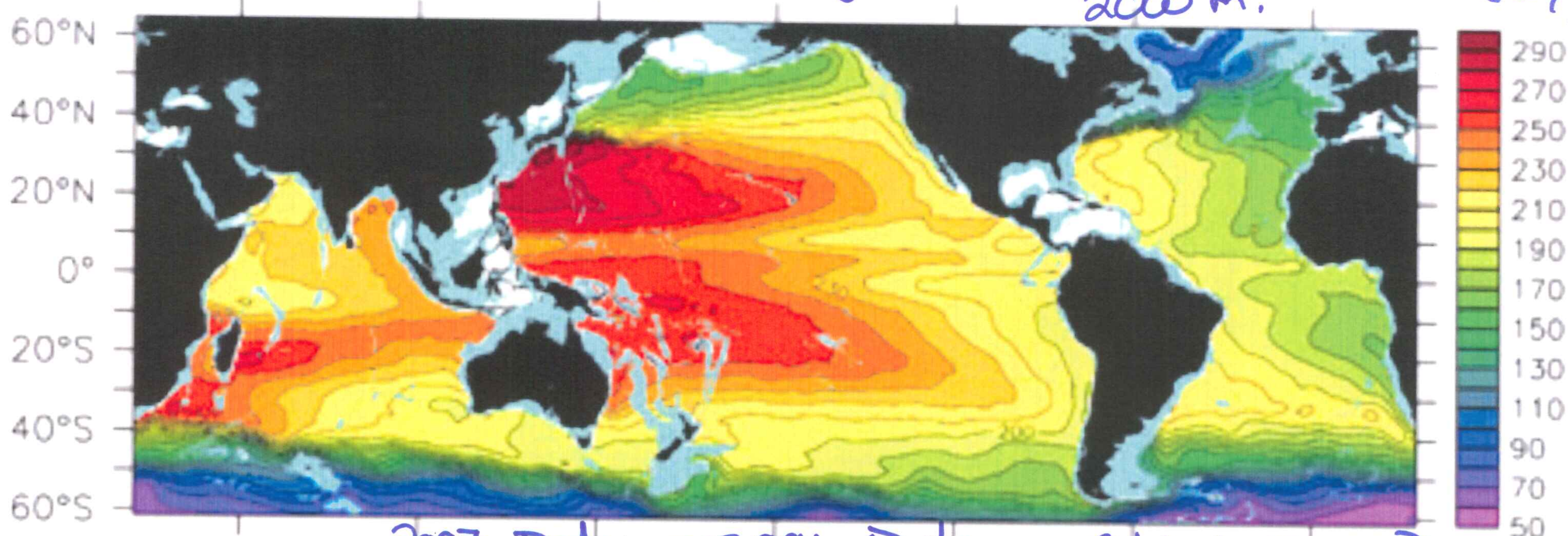
$$\Delta D \equiv \int_{P_1}^{P_2} \delta\, dp \qquad (\text{properly, } \Delta D(P_1, P_2))$$

Next Page $Dyn^{\circ}_{2000}$ for all ocean

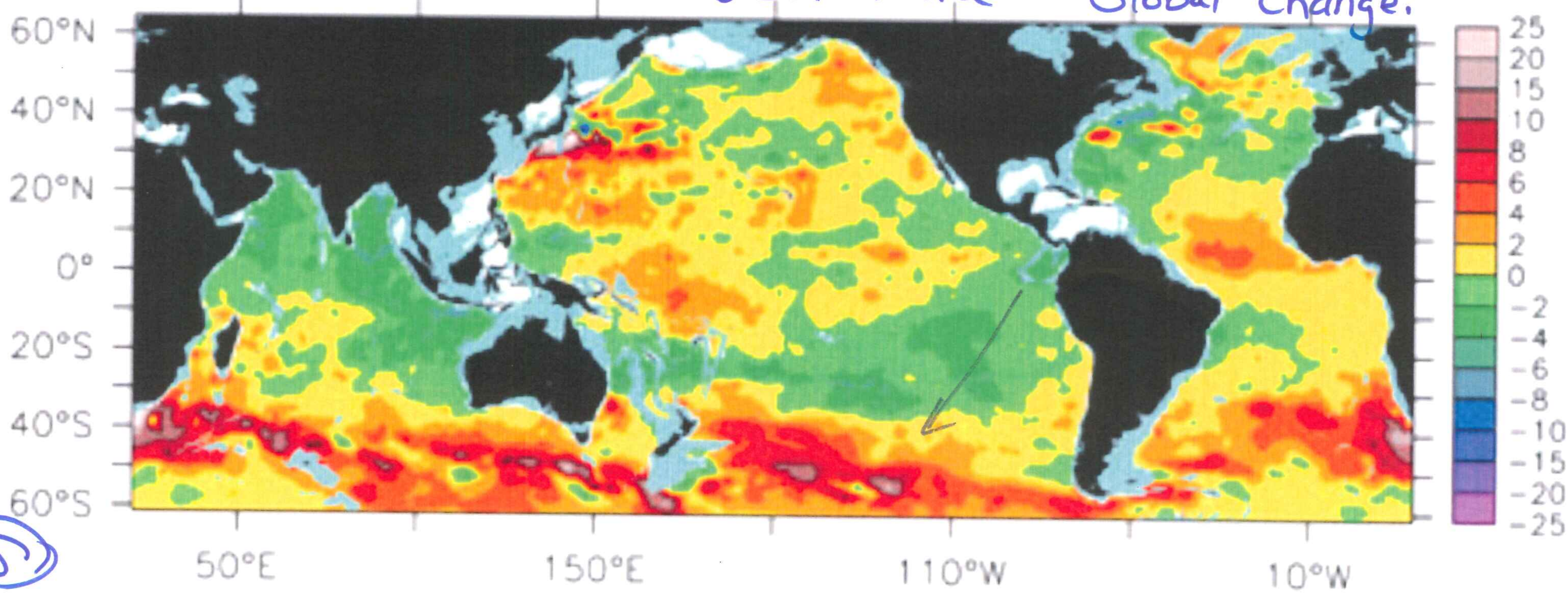
$$\Rightarrow \quad u(P_2) - u(P_1) = \frac{1}{f}\frac{\partial}{\partial y}(\Delta D)$$

$$v(P_2) - v(P_1) = -\frac{1}{f}\frac{\partial}{\partial x}(\Delta D)$$

Mean Dyn Height Relative to 2000 m. $(cm^2/s^2)$




2007 Data - 2001 Data - Global Change?




6

Let's call $\psi(p_1, p_2, x, y) \equiv \frac{1}{f_0}\Delta D$ (where $f \equiv f_0$)
↳ const.

Then $u(p_2) - u(p_1) = \frac{\partial \psi}{\partial y}$

$v(p_2) - v(p_1) = -\frac{\partial \psi}{\partial x}$

$\psi$ is a stream function, a single function that we can use to obtain everything we want about the flow field.

But what about units?

$[\Delta D] = [f][y][u] = \frac{1}{s} \times m \times \frac{m}{s} = \frac{m^2}{s^2}$ ?!

the "steric height" $[\Phi] = m$

well $\Delta D = g\Phi$

but $\psi$'s form resembles the geostrophic stream fcn. (we shall see later)

## Interpretation of Dynamic Height

Clearly $h = \frac{\Delta D}{g}$ is the vertical distance between two pressure surfaces. This means something (see plots) and if we know the long-term shape of some pressure surfaces we can diagnose sub-surface currents.

Let's go one step further and assume we have a fabled "height of 0 velocity", and look at the $\Delta D$ from the surface

Then $v(P_0) = \frac{-1}{f}\frac{\partial}{\partial x}(hg)$

$u(P_0) = \frac{1}{f}\frac{\partial}{\partial y}(hg)$

and $h$ is therefore the surface height required to balance the pressure gradient at the depth of 0 motion.

---

E.G. across Atlantic

$\Delta x \sim 3\times10^3\ \text{km} \sim 3\times10^6\ \text{m}$

$u \sim 10\text{cm/s} \sim 10^{-1}\ \text{m/s}$

$g \sim 10\ \text{m/s}^2$

$f \sim 1.5\,10^{-4}\ 1/\text{s}$

$\Rightarrow 10^{-1} = \frac{1.5\,10^{4}}{3\times10^{6}}\,10\,(h)$

$\Rightarrow 2\ \text{m} = h$

<u>Problem ?</u>